

# Corrugation crack front waves

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## Abstract

The paper presents a model of a dynamic crack with a wavy surface. So far, theoretical analysis of crack front waves has been performed only for in-plane perturbations of the crack front. In the present paper, generalisation is given to a more general three-dimensional perturbation, and equations that govern corrugation crack front waves are derived and analysed.

**Keywords:** Dynamic fracture, crack front waves, asymptotic analysis.

## 1 Introduction

The paper analyses singular fields around a dynamic crack whose surface is slightly perturbed from the original plane configuration. Crack front waves in the plane of the crack were discovered numerically by Morrissey and Rice in [1], and later confirmed analytically by Ramanathan and Fisher [2], using the results of the perturbation analysis of Willis and Movchan [3]. Experimental observations of persistent crack front waves were reported by Sharon, Cohen and Fineberg [4]. The more general development of Willis and Movchan [5] and Woolfries *et al.* [6] extended the analysis to a crack propagating through a viscoelastic medium. The perturbation formulae for the stress intensity factors, specialised to a plane strain formulation, have been used by Obrezanova *et al.* [7] in the stability analysis of rectilinear propagation. A quasi-static advance of a tunnel crack under a mixed mode loading has been analysed by Lazarus and Leblond [8].

The aim of the present paper is to develop a model describing corrugation (out-of-plane) waves along the front of a moving crack. This work is based

on the ideas of the earlier publication by Willis [9]. The plan of the paper is as follows. We begin, in Section 2, with the description of the geometry, governing equations and perturbation functions. A summary of the first-order approximations for the stress intensity factors is presented in Section 2.2. Section 3 includes the study of the corrugation waves in the first-order asymptotic approximation for a basic Mode I loading. In Section 4, we derive the dispersion equation for crack front waves in the mixed mode I-III loading. The technical appendix contains an outline of the fundamental integral identity, and the expressions for effective tractions.

## 2 Basic perturbation formulae

For a linearly elastic medium, we consider a semi-infinite crack with a slightly perturbed surface. The unperturbed configuration of the crack at time  $t$  is defined by

$$S_0(t) = \{\mathbf{x} : -\infty < x_1 < Vt, -\infty < x_2 < \infty, x_3 = 0\}, \quad (1)$$

where  $V$  is a constant crack speed, which does not exceed the Rayleigh wave speed. The perturbation is introduced through deviations of the crack front in both in-plane and out-of-plane directions. The perturbed surface of the crack at time  $t$  is

$$S_\varepsilon(t) = \{\mathbf{x} : -\infty < x_1 < Vt + \varepsilon\varphi(x_2, t), \\ -\infty < x_2 < \infty, x_3 = \varepsilon\psi(x_1 - Vt, x_2)\}. \quad (2)$$

The functions  $\varphi$  and  $\psi$  are smooth and bounded, and  $\varepsilon$  is a small non-dimensional parameter,  $0 \leq \varepsilon \ll 1$ . It is helpful to use the moving-frame coordinates, so that  $X = x_1 - Vt$ .

It is assumed that the medium is loaded so that a stress  $\boldsymbol{\sigma}^{\text{nc}}$  and a displacement  $\mathbf{u}^{\text{nc}}$  would be generated in the absence of the crack. The crack induces additional fields  $\boldsymbol{\sigma}$ ,  $\mathbf{u}$ . They satisfy the equations of motion and the traction boundary conditions on the crack faces:

$$\sigma_{ij,j} - \rho\ddot{u}_i = 0, \quad i = 1, 2, 3, \quad \text{outside the crack} \quad (3)$$

and

$$\sigma_{ij}n_j + \sigma_{ij}^{\text{nc}}n_j = 0, \quad \text{on the crack faces}, \quad (4)$$

and correspond to waves outgoing from the crack as  $x_3 \rightarrow \pm\infty$ .

## 2.1 Local coordinates and asymptotics for stresses

At a point  $\mathbf{x}^0 = (x_1^0, x_2^0, x_3^0)$ , which is on the crack edge at time  $t$ , so that

$$x_1^0 = Vt + \varepsilon\varphi(x_2^0, t), \quad x_3^0 = \varepsilon\psi(x_1^0 - Vt, x_2^0),$$

we define a coordinate system such that

$$\mathbf{x} - \mathbf{x}^0 = \sum_{i=1}^3 x'_i \mathbf{e}'_i,$$

where

$$\begin{pmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \\ \mathbf{e}'_3 \end{pmatrix} = \left\{ \mathbf{I} + \varepsilon \begin{pmatrix} 0 & -\varphi_{,2} & \psi_{,1}^* \\ \varphi_{,2} & 0 & \psi_{,2}^* \\ -\psi_{,1}^* & -\psi_{,2}^* & 0 \end{pmatrix} \right\} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}. \quad (5)$$

Here  $\psi^*$  denotes  $\psi$  evaluated for  $x_1 = Vt$ . The above transformation involves a shift to the crack edge and a further rotation of coordinate axes.

In the new frame, the stress components  $(\sigma'_{i3})$  have the asymptotic form

$$\begin{aligned} \sigma'_{i3}(x'_1, x'_2, 0) &\sim (K_i^{(0)} + \varepsilon K_i^{(1)})/(2\pi x'_1)^{1/2} - (P_i^{(0)} + \Delta P_i) \\ &+ (A_i^{(0)} + \Delta A_i)(x'_1)^{1/2} - (F_i^{(0)} + \Delta F_i)x'_1 + (N_i^{(0)} + \Delta N_i)(x'_1)^{3/2}, \quad i = 1, 2, 3. \end{aligned} \quad (6)$$

The first-order asymptotic approximation of stress-intensity factors was constructed and studied in [3], [10], [11], [12]. In Appendix we include a description of the fundamental identity, which is essential for this work. We also require the dynamic crack face weight function  $[\mathbf{U}]$ , as defined in Appendix. The field  $[\mathbf{U}]$  has a singularity proportional to  $X^{-1/2}H(X)\delta(x_2)\delta(t)$  as  $X \rightarrow 0$ .

## 2.2 First-order perturbations of the stress intensity factors

We begin with the first-order approximation for the stress intensity factors, when

$$K_j \sim K_j^{(0)} + \varepsilon K_j^{(1)}, \quad j = I, II, III. \quad (7)$$

For the Mode-I unperturbed case,  $K_{II}^{(0)} = K_{III}^{(0)} = 0$ , and the perturbation terms are defined by (see [11], [12])

$$K_{II}^{(1)} = -Q_{11} * \psi^* \Theta_{13} K_I^{(0)} - \psi_{,1}^* \omega_{13} K_I^{(0)} - \psi^* \left( \Sigma_{11} + \frac{V^2}{2b^2} \Sigma_{12} \right) A_3^{(0)} \sqrt{\frac{\pi}{2}}$$

$$+ [U]_{11} * \langle P_1^{(1)} \rangle + [U]_{21} * \langle P_2^{(1)} \rangle - \langle U \rangle_{31} * [P_3^{(1)}], \quad (8)$$

$$K_{III}^{(1)} = -Q_{12} * \psi^* \Theta_{13} K_I^{(0)} - \psi_{,2}^* \omega_{23} K_I^{(0)} \\ + [U]_{12} * \langle P_1^{(1)} \rangle + [U]_{22} * \langle P_2^{(1)} \rangle - \langle U \rangle_{32} * [P_3^{(1)}], \quad (9)$$

$$K_I^{(1)} = Q_{33} * \varphi K_I^{(0)} + \left( \frac{\pi}{2} \right)^{1/2} \varphi A_3^{(0)} - \langle U \rangle_{13} * [P_1^{(1)}] \\ - \langle U \rangle_{23} * [P_2^{(1)}] + [U]_{33} * \langle P_3^{(1)} \rangle. \quad (10)$$

The matrix  $\mathbf{Q}$  is a block-diagonal matrix defined in [11]; other functions that appear in the above equations are

$$\Theta_{13} = \Sigma_{11} + \frac{V^2}{2b^2} \Sigma_{12}, \quad \omega_{13} = \frac{\alpha - \beta}{R(V)} (1 + \beta^2)(\alpha + 2\beta) - 2, \\ \omega_{23} = \frac{2\nu}{R(V)} (1 + \beta^2)(\alpha^2 - \beta^2) - 1, \\ \Sigma_{11} = -\frac{4\alpha\beta - (1 + 2\alpha^2 - \beta^2)(1 + \beta^2)}{R(V)}, \quad \Sigma_{12} = \frac{-2(1 + \beta^2 - 2\alpha\beta)}{R(V)}, \\ \alpha^2 = 1 - V^2/a^2, \quad \beta^2 = 1 - V^2/b^2, \quad R(V) = 4\alpha\beta - (1 + \beta^2)^2. \quad (11)$$

Here,  $a$  and  $b$  denote the speeds of longitudinal and shear waves, respectively. The representations for the effective tractions  $P_i^{(1)}, i = 1, 2, 3$ , are given in Appendix.

### 2.3 Crack front waves confined to the plane $x_3 = 0$

Assuming that the out-of-plane deflection is not present ( $\psi = 0$ ), we consider a first-order in-plane perturbation of the crack front and loading in Mode I, so that  $\sigma_{13}^{\text{nc}} = \sigma_{23}^{\text{nc}} = 0$  on the plane  $x_3 = 0$ . In this special case, the only non-zero stress intensity factor is  $K_I$ , and the corresponding perturbation formula reduces to

$$K_I^{(1)} = Q_{33} * \varphi K_I^{(0)} + \left( \frac{\pi}{2} \right)^{1/2} \varphi A_3^{(0)}. \quad (12)$$

According to the Griffith energy balance equation, the energy flux  $\mathcal{G}$  into the crack edge is constant, denoted here by  $\mathcal{G}_c$ :

$$\mathcal{G} \equiv \frac{1 - \nu^2}{E} f_I(v) K_I^2 = \mathcal{G}_c. \quad (13)$$

Here,  $v$  is the local crack speed (to the first-order approximation,  $v = V + \varepsilon \dot{\varphi}$ ) and  $f_I(v)$  is a known function (e.g., [13]):

$$f_I(v) = \frac{v^2 \alpha(v)}{(1 - \nu) b^2 R(v)}. \quad (14)$$

Expanding the Griffith energy balance equation (13) to order  $\varepsilon$ , we obtain

$$2Q_{33} * \varphi + \frac{f'_I(V)}{f_I(V)} \dot{\varphi} + 2m\varphi = 0, \quad (15)$$

where  $m = (\pi/2)^{1/2} A_3^{(0)} / K_I^{(0)}$ . Applying the Fourier transform with respect to  $t$  and  $x_2$  we deduce that a non-zero solution is possible only if the dispersion relation

$$2\overline{Q}_{33}(\omega, k) - i\omega \frac{f'_I(V)}{f_I(V)} + 2m = 0 \quad (16)$$

is satisfied. Here, the Fourier transform  $\overline{Q}_{33}$  is a homogeneous function of degree 1 in  $(\omega, k)$ . At high frequency and large wavenumber, the third term in the above equation can be neglected. Such an equation can be solved for  $\omega/k$ , and a real root represents a speed of wave propagating along the crack front. This computation was performed by Ramanathan and Fisher [2].

### 3 Corrugation waves for a Mode-I basic loading. First-order analysis.

Can a Mode-I basic loading generate a corrugation wave propagating along the crack front? This case corresponds to a non-zero out-of-plane perturbation characterised by the function  $\psi(x_1 - Vt, x_2)$ . Crack stability with respect to out-of-plane deflections can be studied, once a fracture criterion is identified.

If we suppose that  $K_{II} = 0$  then, to lowest order,  $\psi$  must satisfy  $K_{II}^{(1)}(\psi) = 0$ , where  $K_{II}^{(1)}$  is given by (8). The proposition that the crack propagates so as to maintain  $K_{II} = 0$  together with the Griffith energy balance has recently received theoretical support, on the basis of a version of Hamilton's principle [14].

Assuming that the in-plane perturbation of the crack front equals zero, we look into stability against out-of-plane deflections. It is also assumed

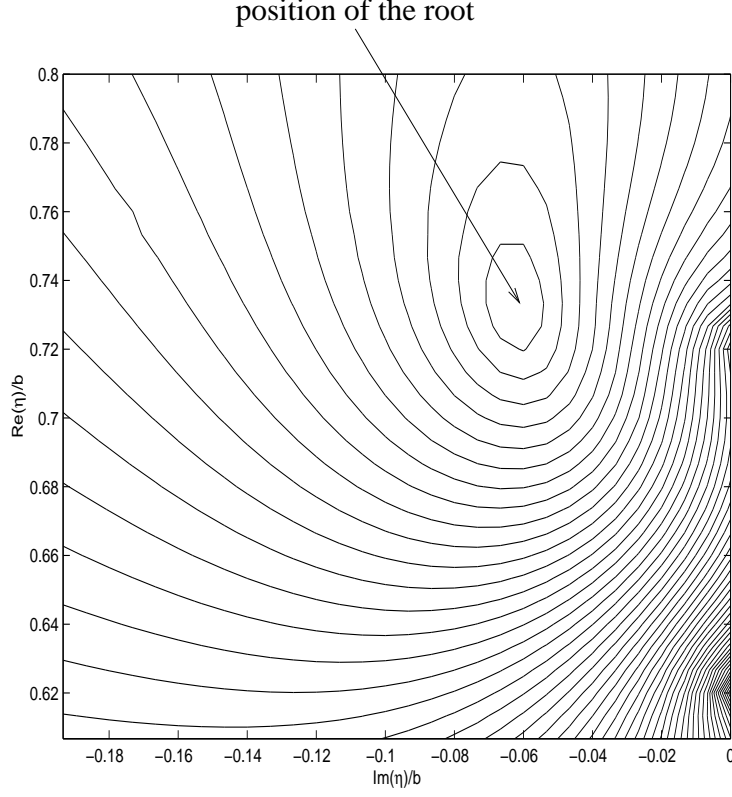


Figure 1: Level curves of the function  $\mathcal{W} = |\overline{Q}_{11}\Theta_{13} - i(\omega/V)\omega_{13}|$ , for  $V/b = 0.69$  and  $\nu = 0.3$ .

that  $\omega \equiv k_1 V$  and  $k_2$  are large. The leading-order approximation of the stress intensity factor  $K_{II}$  yields

$$\overline{K}_{II}^{(1)} = \{-\overline{Q}_{11}\Theta_{13} + i(\omega/V)\omega_{13}\}K_I^{(0)}\overline{\psi}^* = 0. \quad (17)$$

This relation is homogeneous of degree 1 in  $\omega$  and  $k_2$ , and so is non-dispersive.

The numerical study of equation (17) produced the following results.

- For crack speeds  $V$  greater than a critical value  $V_c$  (which is close to 0.6 of the Rayleigh wave speed) there is a value  $\eta = \omega/|k_2|$  with small, negative, imaginary part that satisfies (17). The position of the root is shown in Figure 1; the calculation is produced for the case of  $V/b = 0.69$ , and the diagram shows the level curves of the modulus of the expression in the curly

brackets on the left side of (17).

Figure 1 is accompanied by a three dimensional surface plot, shown in Figure 2, of the function  $\mathcal{W} = |\overline{Q}_{11}\Theta_{13} - i(\omega/V)\omega_{13}|$ ; the surface touches the  $\eta$ -plane at the point corresponding to the root of equation (17).

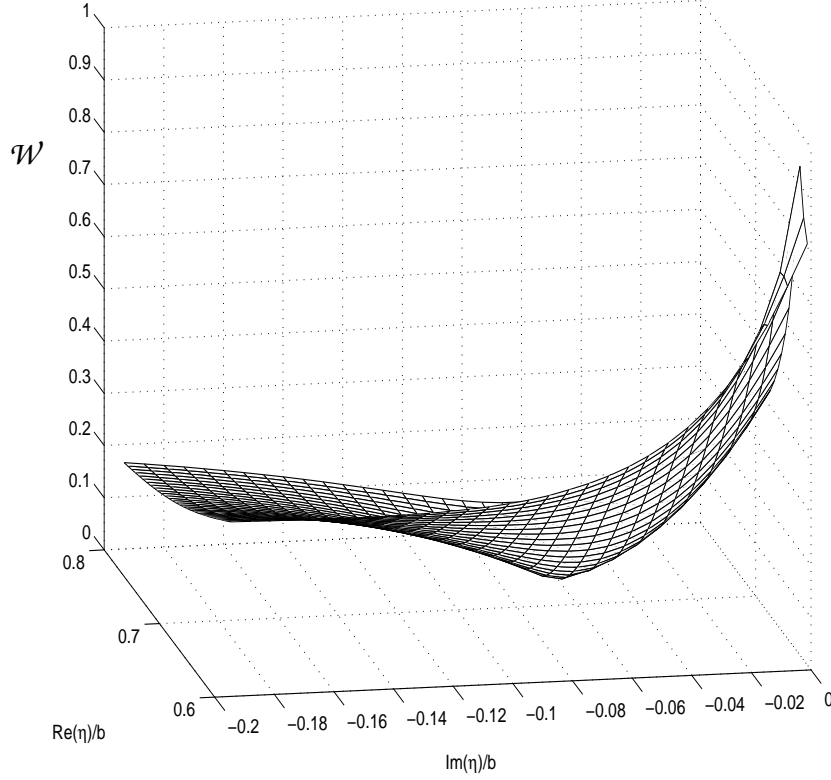


Figure 2: Surface plot of the function  $\mathcal{W} = |\overline{Q}_{11}\Theta_{13} - i(\omega/V)\omega_{13}|$ , for  $V/b = 0.69$  and  $\nu = 0.3$ .

- The "corrugation wave" suffers slow attenuation as it propagates. The imaginary part of  $\eta$ , which characterises the rate of attenuation of the "corrugation wave", is shown in Fig. 3 for different values of the crack front velocity  $V$ , and it decreases with  $V$ .

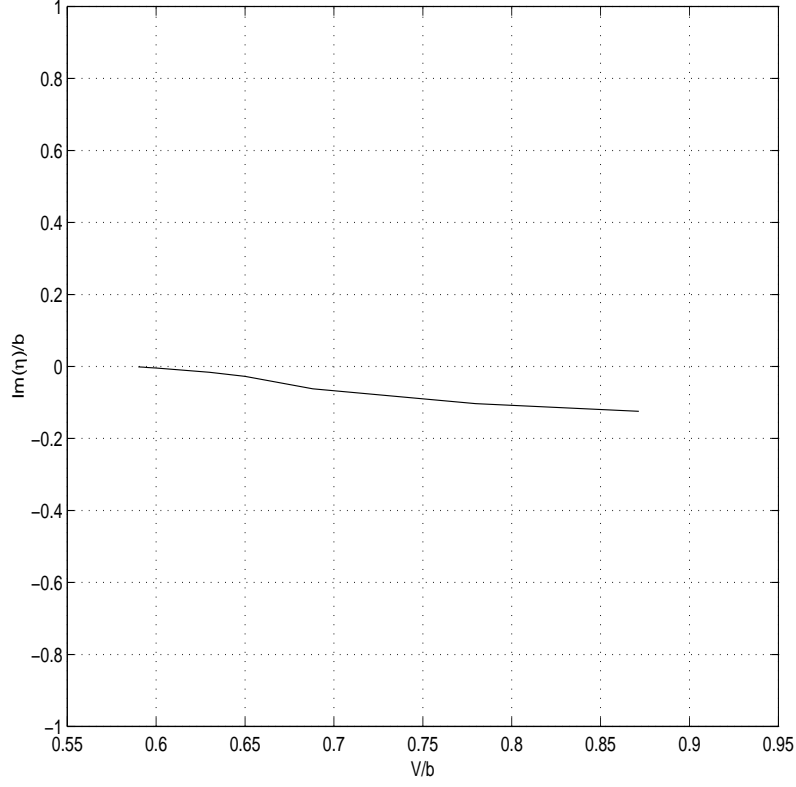


Figure 3: The imaginary part of  $\eta$  as a function of the crack front velocity  $V$ .

#### 4 First-order coupling between in-plane and out-of-plane crack front perturbations for mixed Mode I-III loading

Here, we assume that  $K_{II}^{(0)} = 0$ , whereas  $K_I^{(0)}$  and  $K_{III}^{(0)}$  are non-zero for a half-plane crack propagating with constant speed  $V$  (unperturbed configuration). To first order, the stress intensity factors are represented by the formulae (7), where the perturbation terms  $K_j^{(1)}$ ,  $j = I, II, III$ , are defined by (see [11], [12])

$$K_{II}^{(1)} = -Q_{11} * \psi^* \Theta_{13} K_I^{(0)} - \psi_{,1}^* \omega_{13} K_I^{(0)} - \psi^* \left( \Sigma_{11} + \frac{V^2}{2b^2} \Sigma_{12} \right) A_3^{(0)} \sqrt{\frac{\pi}{2}}$$



$$\begin{aligned}
& +Q_{21} * (\varphi K_{III}^{(0)}) - \varphi_{,2} K_{III}^{(0)} + \sqrt{\frac{\pi}{2}} \varphi A_1^{(0)} \\
& + [U]_{11} * \langle P_1^{(1)} \rangle + [U]_{21} * \langle P_2^{(1)} \rangle - \langle U \rangle_{31} * [P_3^{(1)}], \tag{18}
\end{aligned}$$

$$\begin{aligned}
K_{III}^{(1)} = & -Q_{12} * \psi^* \Theta_{13} K_I^{(0)} - \psi_{,2}^* \omega_{23} K_I^{(0)} + Q_{22} * (\varphi K_{III}^{(0)}) + \sqrt{\frac{\pi}{2}} \varphi A_2^{(0)} \\
& + [U]_{12} * \langle P_1^{(1)} \rangle + [U]_{22} * \langle P_2^{(1)} \rangle - \langle U \rangle_{32} * [P_3^{(1)}], \tag{19}
\end{aligned}$$

$$\begin{aligned}
K_I^{(1)} = & Q_{33} * \varphi K_I^{(0)} + \left(\frac{\pi}{2}\right)^{1/2} \varphi A_3^{(0)} - \psi^* \left(1 - \frac{V^2}{2b^2} \Sigma_{12}\right) A_1^{(0)} \sqrt{\frac{\pi}{2}} \\
& - 2\psi_{,2}^* K_{III}^{(0)} - \langle U \rangle_{13} * [P_1^{(1)}] - \langle U \rangle_{23} * [P_2^{(1)}] + [U]_{33} * \langle P_3^{(1)} \rangle. \tag{20}
\end{aligned}$$

We shall use the criterion of local symmetry  $K_{II} = 0$ , together with the Griffith energy balance equation

$$\mathcal{G} \equiv (2\mu)^{-1} f_I(v) K_I^2 + (2\mu)^{-1} f_{III}(v) K_{III}^2 = \mathcal{G}_c = \text{const.} \tag{21}$$

Taking into account that, to first order,  $v \sim V + \varepsilon \dot{\varphi}$ , we deduce

$$\begin{aligned}
\mathcal{G} = & (2\mu)^{-1} f_I(V) (K_I^{(0)})^2 + (2\mu)^{-1} f_{III}(V) (K_{III}^{(0)})^2 \\
& + \varepsilon (2\mu)^{-1} \left( \dot{\varphi} f_I'(V) (K_I^{(0)})^2 + 2f_I(V) K_I^{(0)} K_I^{(1)} \right. \\
& \left. + \dot{\varphi} f_{III}'(V) (K_{III}^{(0)})^2 + 2f_{III}(V) K_{III}^{(0)} K_{III}^{(1)} \right) + O(\varepsilon^2). \tag{22}
\end{aligned}$$

It follows from (21), (22) and the local symmetry criterion  $K_{II} = 0$  that

$$\begin{aligned}
& \dot{\varphi} (f_I'(V) (K_I^{(0)})^2 + f_{III}'(V) (K_{III}^{(0)})^2) + 2f_I(V) K_I^{(0)} K_I^{(1)}(\varphi, \psi) \\
& + 2f_{III}(V) K_{III}^{(0)} K_{III}^{(1)}(\varphi, \psi) = 0, \tag{23}
\end{aligned}$$

$$K_{II}^{(1)}(\varphi, \psi) = 0. \tag{24}$$

The above equations define the coupling between the in-plane and out-of-plane perturbations of the crack front.

Applying the Fourier transform with respect to  $t$  and  $x_2$  and assuming that  $\omega = k_1 V$  and  $k_2$  are large, we deduce

$$\begin{aligned}
& \overline{\varphi} \left\{ \left( 2f_I(V) \overline{Q}_{33} - i\omega f_I'(V) \right) (K_I^{(0)})^2 + \left( 2f_{III}(V) \overline{Q}_{22} - i\omega f_{III}'(V) \right) (K_{III}^{(0)})^2 \right\} \\
& + \overline{\psi^*} K_I^{(0)} K_{III}^{(0)} \left\{ 2f_{III}(V) \left( -\overline{Q}_{12} \Theta_{13} + i k_2 \omega_{23} \right) + 4i k_2 f_I(V) \right\} = 0, \tag{25}
\end{aligned}$$

$$\{-\overline{Q}_{11}\Theta_{13} + i(\omega/V)\omega_{13}\}\overline{\psi}^*K_I^{(0)} + (\overline{Q}_{21} + ik_2)\overline{\varphi}K_{III}^{(0)} = 0. \quad (26)$$

The system (25), (26) is linear in  $\overline{\varphi}$  and  $\overline{\psi}^*$ , and it possesses a nontrivial solution if and only if the matrix of this system is degenerate. This yields the following dispersion relation:

$$\begin{aligned} &\left\{-\overline{Q}_{11}\Theta_{13} + i(\omega/V)\omega_{13}\right\}\left\{2f_I(V)\overline{Q}_{33} - i\omega f'_I(V) + \left(2f_{III}(V)\overline{Q}_{22} - i\omega f'_{III}(V)\right)\mathcal{K}_0^2\right\} \\ &- \mathcal{K}_0^2(\overline{Q}_{21} + ik_2)\left\{2f_{III}(V)\left(-\overline{Q}_{12}\Theta_{13} + ik_2\omega_{23}\right) + 4ik_2f_I(V)\right\} = 0. \end{aligned} \quad (27)$$

Here  $\mathcal{K}_0 = K_{III}^{(0)}/K_I^{(0)}$ . The above dispersion equation, connecting  $\omega$  and  $k_2$ , is to be analysed numerically to identify possible crack front waves associated with the external mixed mode I-III load.

## Appendix. Fundamental identity and effective tractions.

Here, we briefly describe the method developed in [3], [10], [11]. We use the relation

$$\mathbf{u} = -\mathbf{G} * \boldsymbol{\sigma}, \quad (A1)$$

where  $\mathbf{u}$  and  $\boldsymbol{\sigma}$  denote the values of the displacement vector ( $u_i$ ) and the traction vector ( $\sigma_{i3}$ ) on the surface  $x_3 = 0$  of the half-space  $x_3 > 0$ ;  $\mathbf{G}$  is the Green's matrix function. The symbol  $*$  denotes convolution over  $x_1$ ,  $x_2$  and  $t$ . It is assumed that all waves emanate from the surface  $x_3 = 0$ . A similar identity applies to the half-space  $x_3 < 0$ , with  $\mathbf{G}$  being replaced by  $-\mathbf{G}^T$ .

Three column vectors like  $\mathbf{u}$  can be written side by side to form a matrix  $\mathbf{U}(+0)$ , and similarly  $\boldsymbol{\Sigma}(+0)$  represents the matrix formed from the three corresponding vectors  $\boldsymbol{\sigma}$ . Then

$$\mathbf{U}(+0) = -\mathbf{G} * \boldsymbol{\Sigma}(+0). \quad (A2)$$

The argument  $(+0)$  signifies values on the boundary of the upper half-space. Applying similar reasoning to the identity for the lower half-space  $x_3 < 0$  gives

$$\mathbf{U}(-0) = \mathbf{G}^T * \boldsymbol{\Sigma}(-0). \quad (A3)$$

Next, we note that

$$\{\mathbf{U}(+0)\}^T * \boldsymbol{\sigma}(-0) = -\{\boldsymbol{\Sigma}(+0)\}^T * \mathbf{G}^T * \boldsymbol{\sigma}(-0)$$

$$= -\{\Sigma(+0)\}^T * \mathbf{u}(-0), \quad (\text{A4})$$

$$\begin{aligned} \{\mathbf{U}(-0)\}^T * \boldsymbol{\sigma}(+0) &= \{\Sigma(-0)\}^T * \mathbf{G}^T * \boldsymbol{\sigma}(+0) \\ &= -\{\Sigma(-0)\}^T * \mathbf{u}(+0). \end{aligned} \quad (\text{A5})$$

Subtracting the second line from the first and rearranging gives the identity

$$[\mathbf{U}]^T * \langle \boldsymbol{\sigma} \rangle - \langle \mathbf{U} \rangle^T * [\boldsymbol{\sigma}] = -[\Sigma]^T * \langle \mathbf{u} \rangle + \langle \Sigma \rangle^T * [\mathbf{u}], \quad (\text{A6})$$

where  $\langle f \rangle = \frac{1}{2}(f(+0) + f(-0))$  and  $[f] = f(+0) - f(-0)$ .

In the moving frame associated with the crack edge, we use the coordinate  $X = x_1 - Vt$ . The operation of convolution survives, with functions regarded as functions of  $X, x_2, t$  and the convolutions taken over these new variables.

For the unperturbed crack problem,

$$[\boldsymbol{\sigma}] \equiv 0, \quad [\mathbf{u}] = 0 \text{ when } X > 0, \quad \boldsymbol{\sigma} \equiv \langle \boldsymbol{\sigma} \rangle = -\boldsymbol{\sigma}^{\text{nc}} \text{ when } X < 0. \quad (\text{A7})$$

We interpret equation (A6) relative to the moving frame, and perform factorizations of the Green's function so that  $\mathbf{U}$  and  $\Sigma$  display the related properties

$$[\Sigma] \equiv 0, \quad [\mathbf{U}] = 0 \text{ when } X < 0, \quad \Sigma \equiv \langle \Sigma \rangle = 0 \text{ when } X > 0. \quad (\text{A8})$$

Equations (A2), (A3) yield

$$[\mathbf{U}] = -(\mathbf{G} + \mathbf{G}^T) * \langle \Sigma \rangle, \quad \langle \mathbf{U} \rangle = -\frac{1}{2}(\mathbf{G} - \mathbf{G}^T) * \langle \Sigma \rangle. \quad (\text{A9})$$

The first of these relations defines a Wiener–Hopf problem; the second then gives  $\langle \mathbf{U} \rangle$  directly. The Wiener–Hopf problem uncouples into two subproblems. One, associated with the opening mode I of the crack, is a scalar problem. It was solved in the case of elasticity in [3], and for a viscoelastic medium in [6]. The remaining problem involves modes II and III, coupled. It was solved in [10].

The field  $[\mathbf{U}]$  has a singularity proportional to  $X^{-1/2}H(X)\delta(x_2)\delta(t)$  as  $X \rightarrow 0$ . With the constant of proportionality chosen as  $(2/\pi)^{1/2}\mathbf{I}$ , we call  $[\mathbf{U}]$  the *dynamic weight function* for the crack problem. With this choice, letting  $X \rightarrow +0$  in the identity (A6) generates

$$\mathbf{K} = \lim_{X \rightarrow +0} \left\{ \langle \mathbf{U} \rangle^T * [\boldsymbol{\sigma}^{(0)}] - [\mathbf{U}]^T * \langle \boldsymbol{\sigma}^{(0)} \rangle \right\}, \quad (\text{A10})$$

where  $\mathbf{K}$  denotes the vector of stress-intensity factors  $(K_{II}, K_{III}, K_I)^T$ . The matrix function  $\langle \mathbf{U} \rangle$  represents a dynamical version of Bueckner's non-symmetric weight function, as described in [15] and [10].

We assume that the unperturbed steady-state crack is subjected to a Mode-I loading, and the unperturbed displacement field is a vector function  $\mathbf{u}^{(0)} = \mathbf{u}^{(0)}(x_1 - Vt, x_2, x_3)$ . We can write the resulting displacement field in the form

$$\mathbf{u} \sim \mathbf{u}^{(0)} + \varepsilon \mathbf{u}^{(1)},$$

where  $\varepsilon$  is a perturbation parameter.

The effective tractions  $P_i^{(1)} := -\sigma_{i3}(\mathbf{u}^{(1)})|_{x_3=0}$ ,  $i = 1, 2, 3$ , have the form (see formula (4.11) of [11])

$$P_i^{(1)} = - \sum_{k=1}^2 (\psi \sigma_{ik}^{(0)})_{,k} + \psi \left( \rho V^2 u_{i,11}^{(0)} - 2\rho V \frac{\partial^2 u_i^{(0)}}{\partial t \partial X} + \rho \frac{\partial^2 u_i^{(0)}}{\partial t^2} \right).$$

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